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LETTER TO THE EDITOR

# Two-dimensional dodecagonal quasilattices

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**Abstract.** Two-dimensional quasiperiodic lattices with 12-fold rotational symmetry are constructed by projecting a four-dimensional hyperhexagonal lattice onto a two-dimensional subspace. The projection method in the present case is greatly simplified by representing the four-dimensional lattice as a complex two-dimensional lattice. By choosing windows with different forms we obtain several quasiperiodic tilings, which have a self-similarity. One of the tilings is quite similar to the pattern of the dodecagonal quasicrystal of a NiCr alloy.

Quasicrystals which have long-range quasiperiodic orders with non-crystallographic point symmetries are of current interest (see for example Levine and Steinhardt 1986). The first quasicrystal confirmed by experiment has icosahedral point symmetry (Shechtman *et al* 1984). Then dodecagonal and decagonal quasicrystals were reported (Ishimasa *et al* 1985, Bendersky 1985). The latter two are quasiperiodic only along a plane but periodic along the normal to the plane. The structure of the decagonal quasicrystals is believed to be related to the Penrose tiling, which is a self-similar decagonal quasiperiodic tiling of the plane (Penrose 1974, de Bruijn 1981).

Three methods of constructing dodecagonal quasiperiodic lattices or tilings in two dimensions have been proposed to present. The first is by projection of a twelve-dimensional hypercubic lattice onto a plane (Gähler and Rhyner 1986). The second is based on a generalised grid method (Stampfli 1986); a dodecagonal quasilattice is constructed by using two honeycomb grids superposed in such a way that their symmetry axes cross by $\pi/6$ (30°). The third, which is also due to Stampfli (1986), is based on the deflation method with respect to a triangular tile and a square one. These three dodecagonal quasilattices (or tilings) belong to different local isomorphism (LI) classes.

Recently, by generalising a method of constructing self-similar quasiperiodic patterns in one dimension we have established a systematic method of constructing self-similar quasilattices in two dimensions on the basis of number theory of complex quadratic fields. We have applied the method to the case of an algebraic field associated with the complex cubic root of 1, i.e. $\omega = \exp(2\pi i/3) = (-1+\sqrt{3}i)/2$, and have succeeded in constructing dodecagonal quasilattices and tilings, some of which are different from those mentioned above. The original formulation was algebraic but the result was found subsequently to be reformulated geometrically. We will present the geometrical one.

The two-dimensional Euclidean plane can be identified with the complex plane; mathematically we have an isomorphism $\boldsymbol{R}^2 \simeq \boldsymbol{C}$ based on bijection $(x, y)$ $(\in \boldsymbol{R}) \leftrightarrow z = x+iy$ $(\in \boldsymbol{C})$. Then the two-dimensional triangular lattice with unit lattice spacing is identified with $\tilde{\boldsymbol{Z}} = \{n_1 + n_2\omega \mid n_1, n_2 \in \boldsymbol{Z}\}$, i.e. the set of all the Eisenstein integers; an

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Eisenstein integer is a quadratic algebraic integer associated with $\omega$ (see for example Hardy and Wright 1979). Note that $\tilde{\boldsymbol{Z}}$ forms a ring called an integral domain. Note also that $\bar{\omega}$, the complex conjugate of $\omega$, is also an Eisenstein integer because $\bar{\omega}\ (=\omega^2)=-1-\omega$.

Now, let us define a complex two-dimensional lattice by $\tilde{\boldsymbol{Z}}^2\ (=\tilde{\boldsymbol{Z}}\times\tilde{\boldsymbol{Z}})=\{(\nu_1,\nu_2)\,|\,\nu_1,\nu_2\in\tilde{\boldsymbol{Z}}\}$. Since $\boldsymbol{C}^2\simeq\boldsymbol{R}^4$, this complex lattice is essentially a real four-dimensional lattice, which is the direct product of two identical triangular lattices. We will hereafter refer to this lattice as a hyperhexagonal lattice because it contains three-dimensional hexagonal lattices as its hyperlattice planes.

Before proceeding further, we remark that a multiplication of a complex number $\eta$ such that $|\eta|=1$ to complex numbers in $\boldsymbol{C}$ is a one-dimensional unitary transformation resulting in a rotation of $\boldsymbol{R}^2$.

Let us take the following unimodular matrix of integral domain $\tilde{\boldsymbol{Z}}$:

$$M=\begin{pmatrix}1 & -\bar{\omega}\\ 1 & 1\end{pmatrix}.$$

This matrix represents an area-conserving linear mapping of $\boldsymbol{C}^2$ which leaves $\tilde{\boldsymbol{Z}}^2$ (the hyperhexagonal lattice) invariant. It is easily checked that $M^2-2M-\omega I=0$, where $I$ is the $2\times2$ unit matrix. Therefore the eigenvalues of $M$ are the solutions of the equation $\lambda^2-2\lambda-\omega=0$, which is nothing but the secular equation $\det(\lambda I-M)=0$ ($\det M=-\omega$). Let us denote the two eigenvalues by $\tau_+$ and $\tau_-$ ($|\tau_+|>|\tau_-|$). Obviously $\tau_+\tau_-=-\omega$, so that $|\tau_+||\tau_-|=1$ and $|\tau_+|>1>|\tau_-|$. Solving the secular equation, we obtain $\tau_\pm=1\pm\zeta$, where $\zeta=\exp(\pi\mathrm{i}/6)=(\sqrt{3}+\mathrm{i})/2$ ($\zeta^2=-\bar{\omega}$). Note that $\tau_+$ (or $\tau_-$) represents on the complex plane the longer (or shorter) diagonal of 'unit rhombus' whose vertices are at 0, 1, $\zeta$ and $1+\zeta$. The length of the sides of the rhombus is one and the two inner angles are $\pi/6$ (30°) and $5\pi/6$ (150°). It follows that $|\tau_+|=2\cos(\pi/12)=(\sqrt{3}+1)/\sqrt{2}$ ($\simeq1.931\,85$) and $|\tau_-|=|\tau_+|^{-1}=2\sin(\pi/12)=(\sqrt{3}-1)/\sqrt{2}$. On the other hand, $\arg\tau_+=\pi/12$ (15°) and $\arg\tau_-=-5\pi/12$ (−75°). Since $|\tau_+|$ appears as the ratio of the self-similarity of beautiful dodecagonal quasilattices, as will be shown later on, we shall call it the 'platinum ratio' and denote it by $\tau_{\mathrm{p}}$.

By $M$, $\boldsymbol{C}^2$ decomposes into two invariant subspaces which are complex one-dimensional and real two-dimensional. We denote the subspaces corresponding to eigenvalues $\tau_+$ and $\tau_-$ by $\boldsymbol{C}_+$ ($\simeq\boldsymbol{R}^2$) and $\boldsymbol{C}_-$ ($\simeq\boldsymbol{R}^2$), respectively. The two subspaces are orthogonal to each other owing to normality of $M$, i.e. $MM^\dagger=M^\dagger M$. The left eigenvectors (row eigenvectors) corresponding to $\tau_+$ and $\tau_-$ are given by $(1,\zeta)$ and $(1,-\zeta)$, respectively. They are unitary orthogonal to each other. The projections of $z=(z_1,z_2)^{\mathrm{T}}\in\boldsymbol{C}^2$ onto $\boldsymbol{C}_+$ and $\boldsymbol{C}_-$ are given, apart from an unimportant scale factor, by $z_+=z_1+z_2\zeta$ and $z_-=z_1-z_2\zeta$ and the linear mapping $z\to Mz$ decomposes into two mappings $z_+\to\tau_+z_+$ and $z_-\to\tau_-z_-$. Thus $\boldsymbol{C}_+$ is subject to a dilatation by $\tau_{\mathrm{p}}$ and a rotation by $\pi/12$, while $\boldsymbol{C}_-$ is subject to a contraction by $\tau_{\mathrm{p}}^{-1}$ and a rotation.

With the above preparations, we can now write down a quasilattice on $\boldsymbol{C}_+$ as $D(\phi,W)=\{\nu_1+\nu_2\zeta\,|\,(\nu_1,\nu_2)\in\tilde{\boldsymbol{Z}}^2,\ \nu_1-\nu_2\zeta\in\phi+W\}$, where $\phi$ is a complex number representing the 'phase vector' and $W$ is a bounded domain (in $\boldsymbol{C}_+$) representing the window (the possibility of constructing a dodecagonal lattice by a projection of a four-dimensional lattice was noted by Janssen (1986)). Two quasilattices with the same window but with different phase vectors belong to the same LI class. Obviously, we obtain $D(\phi,W)\subset D(\phi,W')$ if $W\subset W'$. In the following discussions, we will use the conventions $\alpha X=\{\alpha z\,|\,z\in X\}$ and $\bar{X}=\{\bar{z}\,|\,z\in X\}$, where $X$ is a subset of $\boldsymbol{C}$ and $\alpha$ is a complex number.

We assume that $\zeta W = W$ and $\bar{W} = W$, so that $W$ has a dodecagonal symmetry whose point group is $D_{12}$, the dodecagonal dihedral group. Now the $2\times 2$ unimodular matrix

$$R = \begin{pmatrix} 0 & -\bar{\omega} \\ 1 & 0 \end{pmatrix}$$

is a unitary matrix, so that it represents an orthogonal transformation transforming the hyperhexagonal lattice to itself. The order of $R$ is twelve, $R^{12} = I$. It commutes with $M$ and unitary transformation, $z \to Rz$, decomposes into $z_+ \to \zeta z_+$ and $z_- \to -\zeta z_-$. Using these properties together with the symmetry of $W$, we can show (see Katz and Duneau 1986) that $\zeta D(\phi, W)$ and $\bar{D}(\phi, W)$ belong to the same LI class as the one for $D(\phi, W)$. Accordingly, $D(\phi, W)$ has a dodecagonal macroscopic symmetry.

Using several properties of $M$ together with the relationships $\tau_- W = \tau_+^{-1} W$ and $\tau_-^{-1} W = \tau_+ W$, we can show also that $\tau_+ D(\phi, W) = D(\tau_- \phi, \tau_+^{-1} W)$ $(\subset D(\tau_- \phi, W))$ and $\tau_+^{-1} D(\phi, W) = D(\tau_-^{-1} \phi, \tau_+ W)$ $(\supset D(\tau_-^{-1} \phi, W))$, which represent the inflation and the deflation rules, respectively (see Katz and Duneau 1986, Gähler 1986). Thus the dodecagonal quasilattice constructed in this letter has a self-similarity characterised by $\tau_+$.

Note that the self-similarity is connected with an improper dilatation because it is accompanied by a rotation by $\arg \tau_+ = \pi/12$ (Stampfli 1986). This is in contrast to the self-similarity of the Penrose tiling or of the icosahedral quasilattice (Katz and Duneau 1986). Note that a double inflation and a subsequent rotation by $-\pi/6$ give rise to a pure dilatation by $\tau_p^2 = 2 + \sqrt{3}$ $(\simeq 3.732\,05)$ because $\tau_+^2 \zeta^{-1} = \tau_p^2$.

We can assign a bond to each pair of 'algebraic neighbours' (Katz and Duneau 1986). Then we obtain a tiling of $C_+$. All the bonds have an equal length (actually the unit length) because $|\zeta| = 1$. A bond takes one of twelve orientations $1, \zeta, \zeta^2, \ldots, \zeta^{11}$. The basic tiles of the tiling are a regular triangle, a square and the unit rhombus, which can assume twelve orientations.

The form of the window is not completely determined by the symmetry argument alone. The types of vertices contained in a quasiperiodic tiling and the statistics of the vertices are simply determined geometrically from the form of the window (Katz and Duneau 1986). We have investigated several windows with different shapes or different sizes. We will present three of them separately below. In what follows we will denote by $\Delta$ a dodecagonal disc with vertices at $1, \zeta, \zeta^2, \ldots, \zeta^{11}$ and by $H$ a hexagonal disc with vertices at $\rho, \rho\zeta^2, \rho\zeta^4, \ldots, \rho\zeta^{10}$, where $\rho = \tau_+/\sqrt{3}$ with $|\rho| = \tau_p/\sqrt{3}$ $(\simeq 1.115)$ and $\arg \rho = \pi/12$.

(*a*) $W_I = H \cup \zeta H$. The most beautiful tiling among our results is the case where the window is a dodecagonal star given by the union of two hexagons $H$ and $\zeta H$. The resulting tiling is shown in figure 1. A tiling inflated once is also superposed in the same figure. The tiling has four types of vertices with coordinations ranging from four to seven. This is a new dodecagonal tiling. We should emphasise the similarity of this tiling to the schematic pattern drawn from the electron microscope image of the dodecagonal quasicrystal of a NiCr alloy (Ishimasa *et al* 1985).

(*b*) $W_{II} = \Delta$. This window is slightly smaller than $W_I$. The tiling shown in figure 2 does not contain any rhombus but, instead, contains holes of a trigonal hexagon, which is equal to the union of two regular triangles, one square and one unit rhombus. The tiling has only two types of vertices with four and five coordinations.

(*c*) $W_{III} = \rho\Delta$. This window, derived by the projection of a Voronoi polytope of the hyperhexagonal lattice onto $C_-$, is equal to the convex closure of $W_I$. In this case

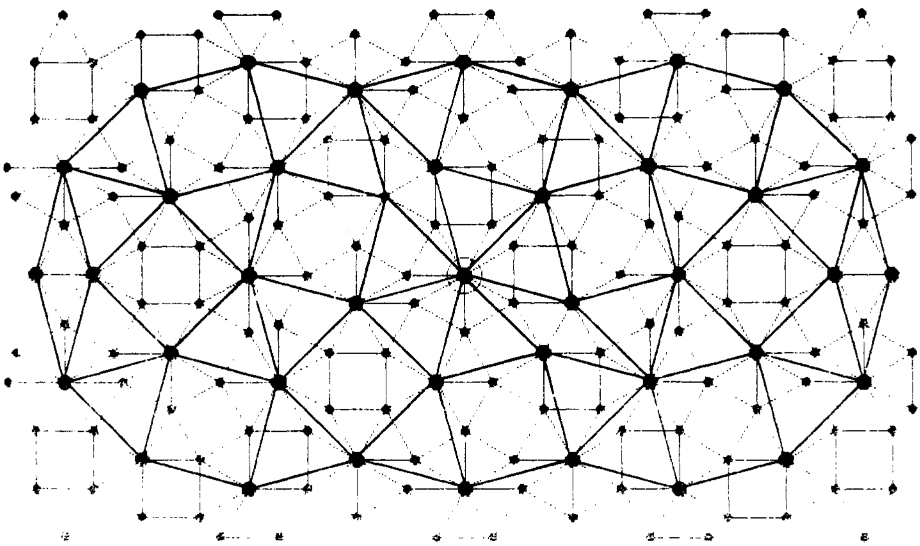

**Figure 1.** The dodecagonal quasiperiodic tiling for the case of a window of a dodecagonal star. Bold lines represent the tiling inflated once.

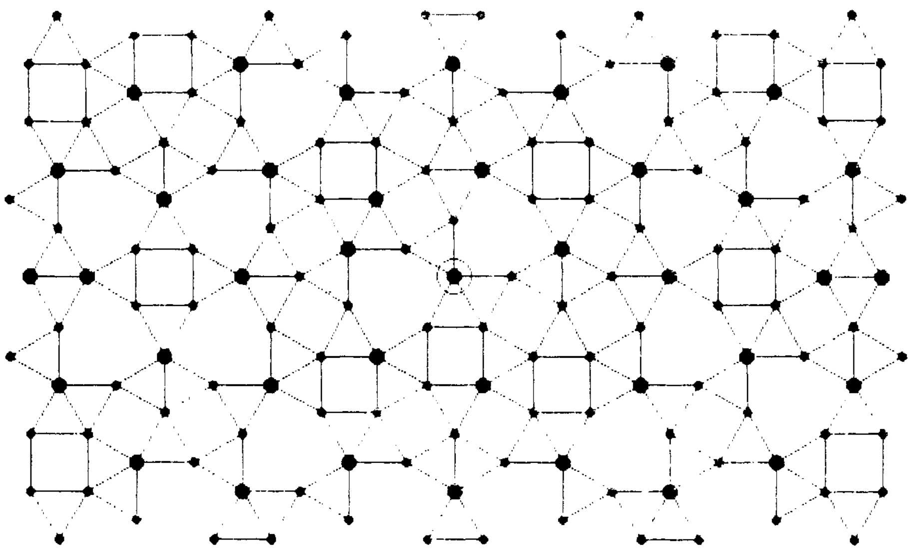

**Figure 2.** A dodecagonal quasiperiodic tiling with holes of a trigonal hexagon.

a bond crosses with another bond because the density of the lattice points is slightly larger than in case I. Fortunately, we can derive a dodecagonal tiling with the same kinds of tiles as in I if we delete systematically a part of the crossing bonds. The resulting tiling can be proved to be identical to the one obtained by Stampfli on the basis of the grid method (see figure 1 in Stampfli (1986)), so that we do not present the figure. This tiling has ten kinds of vertices including the one with a dodecagonal local symmetry.

The Fourier transform of a quasilattice constructed by the projection method can be calculated easily (Zia and Dallas 1985, Katz and Duneau 1986). The intensity of a dodecagonal quasilattice has an exact dodecagonal symmetry. It has also an explicit self-similarity characterised by $\tau_+$. We present the intensity for case I in figure 3. The agreement between the calculated intensity and the diffraction pattern in Ishimasa *et al* (1985) is excellent. Several indistinct 'satellite reflections' in Ishimasa *et al* (1985) can be identified with the ones in figure 3. The details of the dodecagonal tiling will be reported elsewhere.

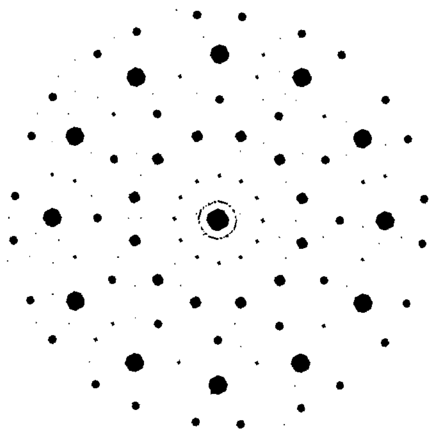

**Figure 3.** The intensity of the Fourier transform of the quasilattice in figure 1. The diameters of the spots are chosen as an increasing function of the intensities.

If we take a different unimodular matrix, we obtain a different quasilattice whose self-similarity is characterised by a root (generally complex) of the corresponding secular equation. The point symmetry is, however, hexagonal and therefore crystallographic. The present method of constructing a quasilattice can be easily generalised to the case of other integral domains of complex quadratic fields. For example, we can construct an octagonal self-similar quasilattice if we use a unimodular matrix associated with Gaussian integers. We can also construct a four-dimensional self-similar quasilattice if we use a quaternian algebraic field. In fact, we have succeeded in constructing a quasilattice with the symmetry of a four-dimensional regular polytope with 120 vertices by using a unimodular matrix associated with Hurwitz integers (Hardy and Wright 1979). This quasilattice contains a three-dimensional icosahedral quasilattice as its hyperlattice plane. These will be published elsewhere.

The work by Stampfli (1986) was pointed out to us by Dr Ishihara of Kyoto University after our work was completed.

## References

Bendersky L 1985 *Phys. Rev. Lett.* **55** 1461-3
de Bruijn N G 1981 *Ned. Akad. Weten. Proc.* A **43** 39-66
Gähler F 1986 *J. Physique* C **3** 115-22
Gähler F and Rhyner J 1986 *J. Phys. A: Math. Gen* **19** 267-77

Hardy G and Wright E M 1979 *An Introduction to the Theory of Numbers* (Oxford: Clarendon) 5th edn
Ishimasa T, Nissen H-U and Fukano Y 1985 *Phys. Rev. Lett.* **55** 511-3
Janssen T 1986 *Acta Crystallogr.* A **42** 261-71
Katz A and Duneau M 1986 *J. Physique* **47** 181-96
Levine D and Steinhardt P J 1986 *Phys. Rev.* B **37** 596-616
Penrose R 1974 *Bull. Inst. Math. Appl.* B **28** 266
Shechtmann D, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951-3
Stampfli P 1986 *Helv. Phys. Acta* **59** 1260-3
Zia R K P and Dallas W J 1985 *J. Phys. A: Math. Gen.* **18** L341-5